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A New Method in Geometry.

By E. Lasker.

INTRODUCTION.

The following lines are based on the theory of moduli, the youngest branch of mathematical science. A method of research in intimate connection with that theory and applicable to algebraical, nay, even analytical formations of any kind, will here be discussed and illustrated. The examples chosen to explain the method, and to show its usefulness, are of a simple nature, and do not require the reader to be acquainted with more of the theory of moduli than is contained in the "Fundamentaltheorem" of Noether.

The method which is the subject of this paper consists in the treatment of formations or configurations of such by means of the syzygetic relations that connect the basic forms of the modulus or moduli corresponding to the configurations. The linear system of such relations of a given order is studied by treating this linear system as an auxiliary space.

The author has used the notation of his "Essay on the geometrical calculus," published in the Proceedings of the London Mathematical Society, 1895 and 1896. This notation may be briefly explained for the plane. If A, B, C are linear forms or points, then ABC denotes their determinant, AB the line joining A and B, and $AB = -BA$. If a, b, c are linear forms subject to contragredient transformations or lines, $a/b/c$ denotes their determinant and a/b the point of intersection of a and b. Any relation between points, such as, say

$$A^2 + 2B^2 = 3C^2 + 9D^2$$

if true, expresses no more nor less than that, if the points-symbols used are simultaneously composed with an arbitrary line ($l = EF$; $Al = AEF$) the relation is true (so that in the above instance

$$(Al)^2 + 2(Bl)^2 = 3(Cl)^2 + 9(Dl)^2).$$

Multiplication is indicated by a dot. $A . B$ denotes, for inst., the product of the points, or linear forms, A and B.

The notation just described is perhaps a trifle simpler and more expressive than the ordinary notation of the invariant calculus, but is, on the whole, very little different from it.

The author has in the examples chosen made use of an auxiliary space, which he has called the λ space. Inasmuch as frequently sets of equations of the type

$$a_1 . u_1 + a_2 . u_2 + \dots\dots = 0$$
$$b_1 . u_1 + b_2 . u_2 + \dots\dots = 0$$

are discussed, he has written them in one line

$$(\lambda_1 . a_1 + \lambda_2 . b_1 + \dots\dots) . u_1 + (\lambda_1 . a_2 + \lambda_2 . b_2 + \dots\dots) . u_2 + \dots\dots = 0$$

and afterwards treated the expressions $\lambda_1 a_1 + \dots\dots$ as linear forms, i. e. points, of the above mentioned λ space. This way of proceeding, though not necessary, seems useful for the purpose of simplifying the calculations that would otherwise be beyond control. It allows, for inst., the advantage of the use of well-known identical relations of the invariant calculus, and it is the distinctive character of the method studied in the paper that the coefficients of these identities are not mere numbers, but forms in the original space, called the x space.

The curve whose equation is $f = 0$ is, in what follows, frequently without further comment denoted with f. This notation seemed almost necessary in a paper where identical relations, such as

$$u_1 . v_1 + u_2 . v_2 + \dots\dots = 0$$

had to be discussed. In any case this way of denoting curves and geometrical formations of any kind has its advantages. It permits to identify an irreducible formation directly with the corresponding prime modulus.

The author's paper twice referred to in what follows, "Zur Theorie der Moduln und Ideale," appeared in the Mathematische Annalen, 1904.

As a first example let a case be considered where one definite syzygetic relation exists between three plane forms. The three forms may be three cubics that have 7 points P_1, P_2 .... P_7 in common, of which we suppose that no 6 of them are on one conic. The system of cubics through $P_1 . . . . P_7$ will be denoted by S and we shall introduce parameters μ_1, μ_2, μ_3 so that any form of S appears in the shape

$$u_1 . \mu_1 + u_2 . \mu_2 + u_3 . \mu_3$$

where u_1, u_2, u_3 are three forms of S that are linearly independent. Finally, let it be agreed upon that the μ_i shall be treated as variables of a μ plane.

Let then a, b, c be three μ points, μ_1, μ_2, μ_3 as well as S being μ lines. Sa, the composition of S with a, is then a definite cubic form in the plane of the variables x_i, the x plane. For inst., $S/\mu_1/\mu_2$ is u_3. If a, b, c are not on one line, Sa, Sb, Sc will be linearly independent and a relation will exist

$$Sa . g_1 + Sb . g_2 + Sc . g_3 = 0$$

where g_1, g_2, g_3 are lines of the x plane, g_1 the line containing the two residual points of the intersection of Sb and Sc and where g_2, g_3 are similarly determined.

If we transform the Sa, Sb, Sc linearly, then a corresponding identity will exist, where the g_1, g_2, g_3 will experience the corresponding cogredient transformation. Hence the g_1, g_2, g_3 may be interpreted as coordinates of a μ point g.

Let $g = g_1 . a + g_2 . b + g_3 . c$, then $Sa . g\,bc + Sb . g\,ca + Sc . g\,ab = 0$ in virtue of the above identity.

This may also be written $Sg = 0$, i. e. the μ line S composed with the μ point g gives zero as result, no matter what the values of the x_i may be; and thus it is put into evidence, that the relation

$$Sa . g\,bc + Sb . g\,ca + Sc . g\,ab = 0$$

will hold good no matter what μ points a, b, c may be chosen.

$g\,ab$ evidently intersects Sa, in virtue of the fundamental relation, besides in the two residual points of its intersection with Sb, also in a point on $g\,ca$. Thus there is a certain x point A on Sa, in which $g\,ab$ intersects that curve no matter how b may be chosen. A is coresidual to the two other points of intersection of $g\,ab$ with Sa, and, these lying on Sb, residual to $P_1 \ldots P_7$.

a being given, A is determined, because Sa is uniquely determined: $g\,ab$ is the line joining the two points A, B corresponding to a and b. In the determination of the μ space there is so much freedom, that we might identify simply a with A, $g\,ab$ with AB. Thus the original identity reads

$$S_A . BC + S_B . CA + S_C . AB = 0$$

$S/_{AB}$ is a μ point, and a pencil of cubics in the x plane. In the above manner of writing, for inst., S/μ_1 is $= u_2 \cdot \mu_2/\mu_1 + u_3 \cdot \mu_3/\mu_1$. The basis of the pencil consists of $P_1 \ldots P_7$ and two points on AB. For, indeed, $S/_{AB}$ contains both S_A and S_B.

The relation between S_A and A is evidently this: A line through A intersects S_A in two points completing with $P_1 \ldots P_7$ the base of a pencil of cubics. In particular, the point that with A and $P_1 \ldots P_7$ completes that configuration, is the intersection of S_A and the tangent to S_A at A. The other 4 tangents from A to S_A, having their point of contact somewhere else, touch S_A in points whose corresponding point coincides with themselves. Through these points P a cubic is possible having P as doublepoint and containing $P_1 \ldots P_7$. The curve of the P is of 6th order and contains $P_1 \ldots P_7$ as doublepoints. It can therefore intersect a cubic through $P_1 \ldots P_7$ only in 4 points besides $P_1 \ldots P_7$. Hence the P points on S_A have been completely identified.

S_{P_i} is the curve of S containing P_i as doublepoint. Indeed, it is according to the fundamental relation

$$S_{P_i} \cdot QR = S_Q \cdot P_i R - S_R \cdot P_i Q$$

and both S_Q and S_R contain P_i.

Let any line l be given. Let to one of its points Q the other Q' be joined that combined with it and $P_1 \ldots P_7$ completes the base of a pencil of cubics. The ∞ straight lines $Q\,Q'$ thus generated will generally belong to a curve of class 3. Indeed if R is an arbitrary point not on l, two points corresponding to each other as Q and Q' will be collinear with R if S_R contains Q and Q'. Hence the points Q on l whose lines $Q\,Q'$ pass through R are the three intersections of l and S_R. If l contains one of the P_i, this reasoning shows that the curve corresponding to l is of class 2. And if l contains two of the P_i, the curve will be a point, namely the point T, belonging to the cubic S_T degenerating into l and the conic through the other 5 points P. This whole reasoning is susceptible of extension to curves of any order, having in the P_i singularities of any kind.

In the geometry of cubics through $P_1 \ldots P_7$ all relations of ordinary plane geometry have their equivalent. This comes from the fact, that the cubics S_A, S_B, $S_C \ldots$ are connected by the same equations as the points A, B, $C \ldots$ themselves. For inst., let A, B, C, D be 4 collinear points such that A and B are harmonically divided by C and D. Then constants α and β will exist, so that

$$C \cdot D = \alpha A^2 + \beta B^2.$$

Hence, since all relations between the points are conserved in the corresponding cubics, we have

$$S_C \,.\, S_D = \alpha\, S_A^2 + \beta\, S_B^2.$$

Interpreting this result for the points $P_1 \ldots . P_7$, Q, R common to S_A, S_B, S_C, S_D, we have the proposition: If A, B, C, D are in harmonic situation, the tangents of S_C and S_D at any one of the P_i, Q, R divide the tangents of S_A and S_B at these points harmonically.

By the same reasoning the connection between the six points that are the intersections of 4 straight lines gives a similar relation between the tangent lines at P_i of the cubics S belonging to the six points. And this principle may be used with every identical relation between points or lines of a plane. The rule is in fact susceptible of yet wider extension as its demonstration makes plain without difficulty.

Wherever between a set of forms a single syzygetic relation exists, as above, the introduction of the μ space is advisable. In what follows we shall however do away with this expedient, in order not to confuse by the introduction of two or more auxiliary spaces. He who can handle operations in various sets of variables with ease will probably be able to shorten much of the work done in what follows. But this capacity is a rare accomplishment. Let u, v, w be forms of the 4th order which have eight points in common and such that no two of them have an infinity of points in common.

The eight points common to $u = 0$, $v = 0$, $w = 0$ may be denoted by P_1, $P_2 \ldots . P_8$ and it is supposed that they do not lie on a conic. The two curves $u = 0$, $v = 0$ will, generally speaking, have 8 more points in common; or, to be accurate, the modulus (u, v) will comprise in all 16 Noetherian conditions. If $f = 0$ is any curve containing P_1, $P_2 \ldots . P_8$, then any form F, such that $f \,.\, F$ belongs to the modulus (u, v), must satisfy 8 conditions, to which we briefly refer as the residual conditions of (u, v). It is not accurate to say that F must contain 8 determinate points in order to satisfy the above relation, because this expresses the truth only when u and v have 16 distinct points of intersection and there may be coincidences. But to simplify the manner of expression we shall assume that u and v are not in contact, and, should in a given case this not be so, we shall understand that the Noetherian conditions of the F modulus will then take the place of the coinciding points. Nor shall this remark be restricted to the case under discussion. In all that follows we shall disregard coincidences of points unless otherwise stated, because the complication thus arising is without influence on the line of reasoning and easy to dissolve by the method of limits, as demonstrated in the paper "Zur Theorie der Moduln und Ideale."

$u = 0$, $v = 0$ will then have 8 residual points in common. These will not lie on a conic, for, from the proposition of Cayley in respect to the intersection of two plane curves it easily follows that, if 8 points of the intersection of two quartics are on a conic then the residual 8 points must likewise be on a conic. And we know that $P_1 \ldots P_8$ are not on a conic.

Let now c_1, c_2 be two forms of third order, such that $c_1 = 0$ and $c_2 = 0$ contain the 8 residual points of $u = 0$, $v = 0$. $w \,.\, c_1 = 0$ and $w \,.\, c_2 = 0$ will then contain all the 16 points of intersection of $u = 0$ and $v = 0$, and therefore two relations will exist

$$u\,a_1 + v\,b_1 + w\,c_1 = 0$$
$$u\,a_2 + v\,b_2 + w\,c_2 = 0$$

where a_1, a_2, b_1, b_2, c_1, c_2 are cubic forms.

Multiplying the first of these equations by c_1, the second by c_2, we obtain by subtraction

$$u\,(a_1 c_2 - a_2 c_1) + v\,(b_1 c_2 - b_2 c_1) = 0,$$

hence

$$a_1 c_2 - a_2 c_1 = - v \,.\, t$$
$$b_1 c_2 - b_2 c_1 = u\,t$$

where t is a form of 2nd order. By eliminating from the two equations u we obtain also

$$b_1 a_2 - a_1 b_2 = w\,t.$$

The interpretation of these equations gives the following results:

From $u\,a_1 + v\,b_1 + w\,c_1 = 0$, as u, v, w have $P_1 \ldots P_8$ in common, it follows that c_1 contains the 8 residual points of (u, v), b_1 those of (u, w), a_1 those of (v, w). c_1 and v have 12 points in common, hence 4 of these points lie on a_1. But c_1 and a_1 have 9 points in common. Consequently a_1, b_1 and c_1 have 5 common points. These, as a look on the three last equations shows, evidently lie on t.

It may equally be inferred that all points common to (a_1, a_2) not on v and w must be on t. Therefore the 9th point of intersection of the two cubics a_1 and a_2 is on t. The same applies to the 9th point of intersection of (b_1, b_2) and (c_1, c_2).

Let us write the two above equations

$$(\lambda_1 a_1 + \lambda_2 a_2)\,u + (\lambda_1 b_1 + \lambda_2 b_2)\,v + (\lambda_1 c_1 + \lambda_2 c_2)\,w = 0$$

where λ_1, λ_2 are indeterminatae. If λ_1, λ_2 are constants, so determined, that $\lambda_1 a_1 + \lambda_2 a_2$ contains point P_1, then $(\lambda_1 a_1 + \lambda_2 a_2)\,u$ will have P_1 as doublepoint. v and w not having contact at P_1, it follows that also $\lambda_1 b_1 + \lambda_2 b_2$ and $\lambda_1 c_1 + \lambda_2 c_2$ must contain P_1. Hence the equation $a u + b v + c w = 0$ is satisfied, if a is the cubic containing the 8 residual points of (v, w) and P_1, b the cubic containing the 8 residual points of (w, u) and P_1, and c is correspondingly determined. a, b, c will then have 5 points in common not on u, v, w, and the conic through them is t. For this construction of t any one of the 8 points $P_1 \dots P_8$ may be utilized. The most general solution of

$$a u + b v + c w = 0$$

where a, b, c are cubics, is then attained by taking an arbitrary point P on t, and constructing a, b, c through their residual 8 points and P.

t is a concomitant of u, v, w which is multiplied by a factor only when u, v, w are subject to a linear transformation, when, for inst., u is replaced by $\alpha u + \beta v + \gamma w$. u, v, w define a linear system S of quartics through $P_1 \dots P_8$ and $t = 0$ is the locus of the point that, with 8 points forming the residual intersection of any two curves of S, completes the configuration of 9 points common to two cubics.

All this may easily be extended to three curves of order n. Thus we may announce: If u, v, w are three curves of nth order having $\frac{1}{2} n (n - 1) + 2$ points P_i in common not situated on a curve of order $n - 2$, then two equations exist

$$u a_1 + v b_1 + w c_1 = 0$$
$$u a_2 + v b_2 + w c_2 = 0$$

where a_1, b_1, c_1, a_2, b_2, c_2 are forms of order $n - 1$. a, b, c have $\frac{1}{2}(n - 1 . n) - 1$ points in common that determine a curve $t = 0$ of order $n - 2$. u, v, w determine a linear system S and any two curves of S intersect in the P_i and in a residual group of $\frac{1}{2} n (n + 1) - 2$ points, which forms part of the base of a pencil of curves of order $n - 1$. The remaining $\frac{(n-2)(n-3)}{2}$ basic points of this pencil are always situate on t.

Further, if u', v', w' are any three linearly independent members of S, the residual pointgroups corresponding to (u', v'), (v', w'), (w', u') are such that triplets of curves of order $(n-1)$ through them and any one of the P_i intersect on $t=0$.

Let us return now to the original case of $n=4$. a_1, a_2 were curves of 3d order through the 8 residual points of (v, w). Let A be the 9th point of intersection of a_1, a_2, and let β and γ be such constants that

$$\gamma v - \beta w$$

contains A. Then this curve will contain all points of intersection of a_1, a_2, hence linear forms p and q will exist such that

$$\gamma v - \beta w = p a_1 + q a_2 .$$

$p a_1 + q a_2$ will therefore contain $P_1 \dots P_8$. Reverting to the argument above, referring to $\lambda_1 a_1 + \lambda_2 a_2$ containing one of the points $P_1 \dots P_8$, it is clear that the same line of reasoning shows $p b_1 + q b_2$ as well as $p c_1 + q c_2$ to contain the points $P_1 \dots P_8$. Hence constants α, α', β', γ' will exist such that

$$p b_1 + q b_2 = \alpha w - \gamma' u$$

and

$$p c_1 + q c_2 = \beta' u - \alpha' v.$$

But identically

$$u(p a_1 + q a_2) + v(p b_1 + q b_2) + w(p c_1 + q c_2) = 0$$

therefore

$$u(\gamma v - \beta w) + v(\alpha w - \gamma' u) + w(\beta' u - \alpha' v) = 0$$

and it follows $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$.

Moreover, since identically

$$\alpha(\gamma v - \beta w) + \beta(\alpha w - \gamma u) + \gamma(\beta u - \alpha v) = 0$$

we have

$$\alpha(p a_1 + q a_2) + \beta(p b_1 + q b_2) + \gamma(p c_1 + q c_2) = 0$$

and

$$\alpha a_1 + \beta b_1 + \gamma c_1 = p \, . \, T$$
$$\alpha a_2 + \beta b_2 + \gamma c_2 = - q \, . \, T$$

where T is a form of 2nd order, which, from the fact that a_1, b_1, c_1 have 5 points in common with $t=0$, can be easily shown to be identical with t.

p and q intersect in the point that together with $P_1 \dots P_8$ is the base of a pencil of cubics. Generally the proposition holds that the quartic which contains the 16 points of the base of a quartic pencil and the 9th point completing with

8 of them the base of a cubic pencil also contains the 9th point completing with the 8 others the base of a cubic pencil.

To show this, let $1\dots\dots8$ and $9\dots\dots16$ be the 16 points common to two quartics; let further I be the point that jointly with $1\dots\dots8$ makes the base of a cubic pencil, and let II be the corresponding point for the set $9\dots\dots16$. The quartic through $1\dots\dots16$ and I may be u, another through $1\dots\dots16$, but neither through I nor II, may be v. Also let A, B be cubics through $1\dots\dots8$ I. Then

$$u = A\alpha - B\beta$$

where α, β are suitably determined linear forms. Also, if p and p' are lines through I

$$\begin{aligned} vp &= A\gamma - B\delta \\ vp' &= A\gamma' - B\delta' \end{aligned}$$

where γ, δ, γ', δ' are forms of 2nd order.

From the two last equations

$$v\,(p\,\delta' - p'\,\delta) = A\,(\gamma\,\delta' - \gamma'\,\delta)$$

hence $$p\,\delta' - p'\,\delta = A\,\varepsilon$$

and $$\gamma\,\delta' - \gamma'\,\delta = v\,\varepsilon$$

also $$p\,\gamma' - p'\,\gamma = B\,\varepsilon$$

where ε is a constant.

We have $u\,\delta - v\,p\,\beta = (\alpha\,\delta - \beta\,\gamma)\,A$.

The points $9\dots\dots16$ lie therefore on $\alpha\,\delta - \beta\,\gamma$ as well as on $\alpha\,\delta' - \beta\gamma'$. Being cubics these forms also contain the point II. But we have identically

$$u\,(\gamma\,\delta' - \gamma'\,\delta) + v\,p\,(\gamma'\,\beta - \delta'\,\alpha) + v\,p'\,(\alpha\,\delta - \beta\,\gamma) = 0$$

hence $$u\,\varepsilon + p\,(\gamma'\,\beta - \delta'\,\alpha) + p'\,(\alpha\,\delta - \beta\,\gamma) = 0$$

u contains therefore II.

Applying this proposition to the equations evolved previously, we conclude that $\alpha w - \gamma u$, $\beta u - \alpha v$, $\gamma v - \beta w$ contain the point P that with $P_1\dots\dots P_8$ completes the base of a cubic pencil. These three quartics evidently belong to one pencil, and the 7 points they have in common besides $P_1\dots\dots P_8\,P$ lie on $t = 0$. To construct t it is therefore only necessary to find the base of the pencil of curves of the system S that contain point P.

The previously deduced laws may be easily extended to forms of higher orders. It is only necessary to interpret the set of identities already made use of for forms of higher orders in order to obtain the corresponding laws concerning them. To give an instance, let in the set of equations used for the demonstration of the proposition concerning a pencil of quartics u be a form of nth order $(n > 4)$. Then we obtain immediately the new proposition: Any form of nth order intersecting a quartic v in $4n$ points and containing one, P, that with 8 of these points completes the base of a cubic pencil, also contains a set of $(n-3)^2$ points that in conjunction with the other $4(n-2)$ points of intersection makes the base of a pencil of $(n-1)^{\text{th}}$ order. In addition it may be shown that this set of $(n-3)^2$ points is the base of a pencil of order $(n-3)$ (namely of the pencil containing α and β).

Let us now attack a case where more than two syzygetic relations obtain, for instance that of three cubics having six points in common. If u, v, w are these cubics, and if the six points $P_1 \dots P_6$ common to them are not on a conic, three relations will exist

$$\begin{aligned} a_1 u + b_1 v + c_1 w &= 0 \\ a_2 u + b_2 v + c_2 w &= 0 \\ a_3 u + b_3 v + c_3 w &= 0 \end{aligned}$$

where the a_i, b_i and c_i are conics. For the existence of a relation

$$a u + b v + c w = 0$$

it is only required that $a = 0$ should contain the three residual points of the intersection of (v, w), and, a being a conic chosen in conformity with this condition, b and c are uniquely determined. Here the number of forms is considerable and, to deal with them efficiently, it is advisable to write the three relations in one line

$$0 = (\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3) u + (\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3) v + (\lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3) w$$

and to interpret λ_1, λ_2, λ_3 as indeterminate coordinates of a point in a λ plane. Finally we write

$$\begin{aligned} \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 &= a \\ \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 &= b \\ \lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3 &= c \end{aligned}$$

so that a, b, c are points in the λ plane and of the 2nd order in the original plane which, for brevity, we designate the x plane.

The equation $au + bv + cw = 0$ shows that the three λ points a, b, c are always collinear. Composing the equation with a, we obtain

$$ab.v + ac.w = 0$$

where ab denotes the matrix composed of a and b and where multiplication is indicated by the point. From this relation it follows that ab is divisible by w and ac by v. A form g will therefore exist such that

$$ab = g.w$$

$$ca = g.v$$

and similarly $$bc = g.u$$

where g is a line in the λ plane and in the x plane is of the 1st order.

Let l be any λ line. Then al, bl, cl, the compositions of the λ points a, b, c with l, are magnitudes in the λ space, and conics in the x plane.

From $al.u + bl.v + cl.w = 0$ it is clear by reasoning analogous to that previously used, that al, bl and w have three, and therefore al, bl and cl have one point in common. This point lies on the line $g/l/m$, m denoting any other λ line. $g/l/m$ contains also the 4th point of intersection of (al, am), (bl, bm), (cl, cm). This fact is put in evidence by the identity

$$al.bm - am.bl = ab/l/m = w.g/l/m$$

al and am have 3 points in common with w, hence their 4th point of intersection lies on $g/l/m$.

Let L be any λ point. gL denotes then a definite x line, and aL, bL, cL pencils of conics. aL always contains the three residual points of (v, w); therefore only the 4th point of the base of the pencil aL is variable. It lies on gL, and the same is true of the 4th point of the pencils bL, cL. Indeed, it is identically

$$aLN.bMN - aMN.bLN = abN.LMN = w.gN.LMN$$

where N is any λ point. Hence aLN and aMN intersect apart from the three points of intersection on w, on gN. Similarly the point common to aLN, bLN, cLN is the point of intersection of the lines gL and gN.

To any point A of the x plane correspond two others B and C in this fashion: Through A a pencil of conics al and am is determined. To it correspond pencils (bl, bm) and (cl, cm), whose 4th point of intersection is B and C respectively. A, B, C are in a straight line, namely $g/l/m$. al, bl, cl and am, bm, cm intersect on the same straight line.

The correspondence between A, B, C is therefore such that if A moves on any conic through the 3 basic points of the system a, B and C will move on related conics, while the line A, B, C will revolve round the point common to the three conics in question.

If A coincides with one of the points P_i, then B and C will also coincide with it. This follows at once from the fundamental identity when it is assumed that a contains P_i.

To two conics of the set a through P_1 correspond two conics of the sets b and c through P_1 and each triplet of them determines a point of intersection. The two points of intersection thus derived and P_1 are collinear.

Hence a certain line p_1 passes through P_1, and similarly 5 other lines $p_2, p_3 \ldots p_6$ pass through P_2, P_3, P_4, P_5, P_6 such that to each conic of the set a through P_i correspond conics of the set b and c through P_i intersecting on p_i. Consequently the conics of the sets a, b, c through P_i and P_j intersect in the point of intersection of p_i and p_j. A very curious net of intersections is thus generated.

The fact that for each position of the λ line l al, bl, cl contain, each, three fixed points, leads to this proposition: Besides g three λ lines α, β, γ exist, which are in the x plane of first order and which composed with the λ points a, b, c give zero. With other words, if a_1, a_2, a_3 are the components of the λ point a two identities exist

$$a_1 g_1 + a_2 g_2 + a_3 g_3 = 0, \qquad a_1 \alpha_1 + a_2 \alpha_2 + a_3 \alpha_3 = 0,$$

where g_1, g_2, g_3 are the coefficients of the λ line g, α_1, α_2, α_3 those of another λ line α. a is then a λ point common to g and α, hence

$$a = g/\alpha, \qquad b = g/\beta, \qquad c = g/\gamma.$$

And from $a u + b v + c w = 0$ it follows

$$(u\alpha + v\beta + w\gamma)/g = 0.$$

$u \,.\, \alpha + v \,.\, \beta + w \,.\, \gamma$ is therefore congruent with g, i. e. a multiple of g. We may put it $= - t \,.\, g$, where t is a form of 3d order in the x plane, a number in the λ plane. Thus we have

$$u \,.\, \alpha + v \,.\, \beta + w \,.\, \gamma + t \,.\, g = 0,$$

and

$$\alpha \beta \gamma = t, \qquad \beta \gamma g = u, \text{ etc.}$$

If L is any λ point, then αL, βL, γL, $g L$ correspond to each other according to the rules of linear transformations, i. e. to any given g line corresponds one α, β, γ line, and vice versa; and to a g line through a given point correspond α, β, γ lines through dependent (correspondent) points.

Now from $g/\alpha = a$ and the identity

$$g L \,.\, \alpha M - g M \,.\, \alpha L = g/a/L M = a L M$$

it is evident that $g L$ and $g M$ as well as $g L$ and αL intersect on $a L M$. Consequently the intersection of $g L$ and $g M$ is again found to be the point common to $a L M$, $b L M$, $c L M$; and the 4th point of intersection of $a L M$ and $a L N$ (formerly called A) is on $g L$ and αL. The triangle of self-corresponding lines of $g L$, αL is evidently that formed by the three points residual to the intersection of (v, w). The linear correspondences of the x plane, characterized by $g L$, αL, βL, γL, may therefore be constructed as follows: Let the three residual triangles of (u, v), (v, w), (w, u) be the self-corresponding ones of three linear transformations; let further to the line p_1 correspond three arbitrarily selected distinct lines α', β', γ' through P_1. Then three correspondences α, β, γ of the plane are thereby determined, such that any line of the plane is cut by its corresponding α line in the point previously called A, etc., and that α, β, γ lines corresponding to the same (g) line intersect in the same point P only, when P is on a cubic $t = 0$ that contains the six points P_1, P_2, P_3, P_4, P_5, P_6.

All this may again be immediately extended to suitably restricted forms of higher orders, for inst. to three quartics having 9 points in common. The matter of generalization becomes simply a question of counting the number of points of intersection, the order numbers of the various curves introduced and the number of constants of these curves. It is not difficult to extent this method to three plane curves of any orders having any number of common points and therefore any number of syzygetic relations. We shall now enlarge the scope of the work by considering the relations of 4 given plane forms.

As a first instance take the case of 4 conics u_1, u_2, u_3, u_4 having no common point and which are linearly independent. Their linear system may be S. Let g_1, g_2, g_3, g_4 be forms of 1st order, then two independent relations of the type

$$g_1 u_1 + g_2 u_2 + g_3 u_3 + g_4 u_4 = 0$$

will exist. That there will be at least 2 follows from the consideration of the cubics through the 8 points (u_1, u_2), (u_3, u_4), which evidently must be expressible in the way $g_1 u_1 + g_2 u_2$ as well as $g_3 u_3 + g_4 u_4$. That there will be no more than two is clear from the fact that these 8 points cannot lie on one conic (or u_1, u_2, u_3, u_4 would be linearly dependent). Hence g_1, g_2, g_3, g_4 will be points on a λ line, or homographics, whose vertices are somewhere in the x plane.

Composing the identity $g_1 . u_1 + g_2 . u_2 + g_3 . u_3 + g_4 . u_4 = 0$ with g_4, we have

$$g_1 g_4 . u_1 + g_2 g_4 . u_2 + g_3 g_4 . u_3 = 0.$$

Hence constants α_1, α_2, α_3 will exist such that

$$\begin{aligned} g_1 g_4 &= \alpha_2 . u_3 - \alpha_3 . u_2 \\ g_2 g_4 &= \alpha_3 . u_1 - \alpha_3 . u_3 \\ g_3 g_4 &= \alpha_1 . u_2 - \alpha_2 . u_1 . \end{aligned}$$

It also follows that

$$\alpha_1 . g_1 g_4 + \alpha_2 . g_2 g_4 + \alpha_3 . g_3 g_4 = 0.$$

$\alpha_1 . g_1 + \alpha_2 . g_2 + \alpha_3 . g_3$ is therefore a numerical multiple of g_4. Consequently a number α_4 exists such that

$$\alpha_1 . g_1 + \alpha_2 . g_2 + \alpha_3 . g_3 + \alpha_4 . g_4 = 0.$$

We shall now interpret these equations. g_1, g_2, g_3, g_4 are homographic pencils of lines through points that will be called A_1, A_2, A_3, A_4. $g_1 g_4$, $g_2 g_4$, $g_3 g_4$ are conics having 4 points in common, namely A_4 and say E, F, G. $g_1 g_4$ contains A_1, A_4 and the 4 points (u_2, u_3). From the identity

$$g_1 g_2 . g_4 l + g_2 g_4 . g_1 l + g_4 g_1 . g_2 l = 0$$

where l is any λ point, it follows that also $g_1 g_2$ contains E, F, G. These three points are therefore common to the six conics $g_1 g_2 \cdots g_3 g_4$.

The pencil of cubics $u_1 . g_1 + u_2 . g_2$ contains the 4 points (u_1, u_2) and as the original identity shows, also the four (u_3, u_4). The pencil passes therefore also through another point $B_{1,2} = B_{3,4}$. Let $B_{1,3} = B_{2,4}$ and $B_{2,3} = B_{1,4}$ be similarly determined. Since $u_1 . g_1 + u_2 . g_2$ contains $B_{1,2}$, also $(u_1 . g_1 + u_2 . g_2) g_2 = u_1 . g_1 g_2$ will. Hence $B_{1,2}$ is situate on both $g_1 g_2$ and $g_3 g_4$. The points of intersection of the six conics $g_1 g_2 \cdots g_3 g_4$ are herewith completely laid down.

$\alpha_1 . g_1 + \alpha_2 . g_2$ is a pencil whose vertex lies on $(\alpha_1 . g_1 + \alpha_2 . g_2) g_2 = \alpha_1 . g_1 g_2$. It also lies on $g_3 g_4$. But its vertex is neither E nor F or G, because it generally does not lie on $g_1 g_3$, which may be most easily shown by the analysis of a particular example. Such an example is most readily obtained by starting inversely from 4 linearly dependent pencils $g_1 \ldots . g_4$ and constructing 4 forms u_1, u_2, u_3, u_4 according to the above identities. It then follows that the vertex of $\alpha_1 \cdot g_1 + \alpha_2 \cdot g_2$ must be $B_{1,2}$. If l is any λ point $\alpha_1 \cdot g_1 l + \alpha_2 \cdot g_2 l$ in virtue of the relation between g_1, g_2, g_3, g_4 is the line joining the intersection of $g_1 l$ and $g_2 l$ with that of $g_3 l$ and $g_4 l$. This line, with varying l, revolves round $B_{1,2}$.

E, F, G remain invariant when u_1, u_2, u_3, u_4 are subject to linear transformations. Starting with *any* 4 forms of S, E, F, G will therefore remain the same. But S contains 4 squares of linear forms and it can be shown without difficulty that E, F, G are the corners of the diagonal triangle of the complete quadrangle of lines whose squares belong to S.

Summarizing, we obtain a proposition as follows: Let 4 conics u_1, u_2, u_3, u_4 not containing a common point nor linearly dependent be arbitrarily given. The 8 points (u_1, u_2) and (u_3, u_4) determine the point $B_{1,2}$ completing the base of a pencil of cubics through them. $B_{1,3}$ and $B_{2,3}$ are similarly constructed. The conic through $B_{1,2}$ and (u_3, u_4) has with that through $B_{1,2}$ and (u_1, u_2) three points E, F, G in common which, with 4 points common to any two conics of the system S, always lie on one conic. The conics through E, F, G and (u_i, u_j) may be denoted $g_i\, g_j$. Then $g_1 g_4$, $g_2 g_4$, $g_3 g_4$ have besides E, F, G another point A_4 in common, etc. Through A_1, A_2, A_3, A_4 a single infinity of lines g_1, g_2, g_3, g_4 will pass whose 6 points of intersection will lie on the 6 above conics $g_1 g_2 \ldots . g_3 g_4$ and the sides of whose diagonal triangles revolve round $B_{1,2}$, $B_{1,3}$, $B_{2,3}$. It also easily follows that each corner of these diagonal triangles moves upon a conic through E, F, G (the point of intersection of the lines (g_1/g_2), $(g_3/g_4) \,|\, (g_1/g_3)$, (g_2/g_4) for inst. moves upon the conics $(\alpha_1 \cdot g_1 + \alpha_2 \cdot g_2)$, $(\alpha_1 \cdot g_1 + \alpha_3 \cdot g_3)$).

The process just made use of, if applied to a set of forms of nth order, will lead to very remarkable results. But the calculation becomes somewhat complex when n is large. The principles involved may however be well explained in the case $n = 3$. Let then u_1, u_2, u_3, u_4 be 4 cubics that are linearly independent, have no point in common and such that through the 18 points (u_1, u_2) and (u_3, u_4) only the minimum number, namely three linearly independent quintics can pass.

Let v_1, v_2, v_3, v_4 be conics, suitably determined, then three independent relations will exist

$$(1) \qquad u_1 \cdot v_1 + u_2 \cdot v_2 + u_3 \cdot v_3 + u_4 \cdot v_4 = 0$$

and the λ space will therefore be a plane. It follows $v_1 v_2 v_3$ is divisible by u_4,

$$v_1 v_2 v_3 = u_4 . w \tag{2}$$

where w is a x form of 3d order, a number in the λ plane. Similarly

$$v_4 v_2 v_3 = - u_1 . w, \text{ etc.} \tag{2}$$

Composing the fundamental identity with v_4, we have

$$u_1 . v_1 v_4 + u_2 . v_2 v_4 + u_3 . v_3 v_4 = 0$$

and therefore forms a_1, a_2, a_3 will exist, such that

$$\begin{aligned} v_1 v_4 &= a_3 . u_2 - a_2 . u_3 \\ v_2 v_4 &= a_1 . u_3 - a_3 . u_1 \\ v_3 v_4 &= a_2 . u_1 - a_1 . u_2 \end{aligned} \tag{3}$$

a_1, a_2, a_3 will be λ lines and x lines.

Composing the first identity with v_1

$$0 = v_1 a_3 . u_2 - v_1 a_2 . u_3.$$

(4) Hence $v_1 a_3 = b_1 . u_3$, $v_1 a_2 = b_1 . u_2$, where b_1 is a number. Similarly

$v_2 a_3 = b_2 . u_3$, $v_2 a_1 = b_2 . u_1$, $v_3 a_1 = b_3 . u_1$, $v_3 a_2 = b_3 . u_2$, $v_4 a_1 = b_4 . u_1 \ldots .$
where b_2, b_3, b_4 are three other numbers.

Composing the original identity with v_1, we have

$$u_2 . v_1 v_2 + u_3 . v_1 v_3 + u_4 . v_1 v_4 = 0,$$

or

$$u_2 . v_1 v_2 + u_3 . v_1 v_3 + u_4 (a_3 u_2 - a_2 u_3) = 0,$$

consequently a λ and x line a_4 will exist such that

$$\begin{aligned} v_1 v_2 &= a_4 . u_3 - a_3 . u_4 \\ v_3 v_1 &= a_4 . u_2 - a_2 . u_4 \end{aligned} \tag{3}$$

and similarly

$$v_2 v_3 = a_4 . u_1 - a_1 . u_4 .$$

Composing $v_1 v_2 = a_4 u_3 - a_3 u_4$ with v_3, we have

$$u_4 w = a_4 v_3 . u_3 - a_3 v_3 . u_4$$

or

$$u_4 w = b_3 u_4 . u_3 - a_3 v_3 . u_4$$

and (5)

$$w = b_3 . u_3 - a_3 v_3 .$$

Composing the original identity with a_3

$$a_3 v_1 . u_1 + a_3 v_2 . u_2 + a_3 v_3 . u_3 + a_3 v_4 . u_4 = 0$$

$$b_1 u_3 . u_1 + b_2 u_3 . u_2 + (b_3 u_3 - w) u_3 + b_4 u_3 . u_4 = 0$$

$$w = b_1 u_1 + b_2 u_2 + b_3 u_3 + b_4 u_4 \tag{6}$$

w therefore belongs to the linear system of u_1, u_2, u_3, u_4.

From the identity

$$v_1 v_2 . v_3 l + v_2 v_3 . v_1 l + v_3 v_1 . v_2 l = v_1 v_2 v_3 . l$$

where l is any λ line, inserting the values above found

$$a_4 . (v_1 l . u_1 + v_2 l . u_2 + v_3 l . u_3) - u_4 . (a_1 . v_1 l + a_2 . v_2 l + a_3 . v_3 l) = - u_4 . W . l.$$

But $v_1 l . u_1 + v_2 l . u_2 + v_3 l . u_3 = - v_4 l . u_4$ owing to the original identity. Consequently

$$a_1 . v_1 l + a_2 . v_2 l + a_3 . v_3 l + a_4 . v_4 l = W . l. \tag{7}$$

Composing this with some point M

$$a_1 M . v_1 l + a_2 M . v_2 l + a_3 M . v_3 l + a_4 M . v_4 l = W . l M$$

or (8) $$a_1 M . v_1 + a_2 M . v_2 + a_3 M . v_3 + a_4 M . v_4 = W . M.$$

Again, identifying l in 7 with a_1 and utilizing (4) and (5)

$$b_1 . a_1 + b_2 . a_2 + b_3 . a_3 + b_4 . a_4 = 0. \tag{9}$$

Hence

$$a_1 . (b_4 . v_1 l - b_1 . v_4 l) + a_2 (b_4 . v_2 l - b_2 . v_4 l) + a_3 (b_4 . v_3 l - b_3 . v_4 l) = W . b_4 l$$

by means of (7) and (9).

But a_1, a_2, a_3 and l considered as λ lines can only be connected by one linear identity. Hence it follows

$$a_1 / a_2 / a_3 = \varepsilon . b_4 . W$$

$$a_1 / a_2 / l = \varepsilon (b_4 . v_3 l - b_3 v_4 l) \text{ or simpler}$$

$$a_1 / a_2 = \varepsilon (b_4 . v_3 - b_3 . v_4) \tag{10}$$

and similarly
$$a_2 / a_3 = \varepsilon (b_4 . v_1 - b_1 . v_4)$$

$$a_3 / a_1 = \varepsilon (b_4 . v_2 - b_2 . v_4)$$

where ε is a numeric constant not only in the λ plane, but also in the x plane, whose value we shall assume to be, for simplicity, $= 1$. Similar relations obviously hold for $a_1 a_2 a_4$, $a_1 a_4$, $a_2 a_4$ and $a_3 a_4$.

Multiplying the original identity by b_4 and inserting the value of W, we obtain

$$(11)\quad u_1 . (b_4 . v_1 - b_1 . v_4) + u_2 . (b_4 . v_2 - b_2 . v_4) + u_3 . (b_4 . v_3 - b_3 . v_4) + W . v_4 = 0$$

These eleven relations are sufficient and necessary to explain the connections existing between the various forms introduced.

u_1, u_2, u_3, u_4 being given, all the other forms are determined. We may ask how far the giving of some of the forms of the set above mentioned determines the whole set. Let $b_1, b_2, b_3, b_4, a_1, a_2, a_3, a_4$ be given in accordance with (9). W is then determined by (10). Of the set v_1, v_2, v_3, v_4 any one may yet be arbitrarily chosen, but then, on account of (10) the whole set is known. After this, on account of (4) and (5), also u_1, u_2, u_3, u_4 are known. $b_1 . u_2$, for inst., is $v_1 a_2$, $b_1 . u_1 = W + a_1 v_1$, $b_1 . v_4 = b_4 . v_1 - a_2/a_3$. The u_1, u_2, u_3, u_4 so found will be connected by

$$u_1 . v_1 + u_2 . v_2 + u_3 . v_3 + u_4 . v_4 = 0.$$

where the v_1, v_2, v_3, v_4 have the above significance, since

$$\begin{aligned} b_1^2 . v_1 v_2 v_3 &= v_1 (b_2 . v_1 - a_3/a_4)(b_3 . v_1 - a_4/a_2) \\ &= v_1 (a_3/a_4)(a_4/a_2) = v_1 a_4 . a_3 a_4 a_2 = b_1 u_4 . b_1 W \end{aligned}$$

and $v_1 v_2 v_3 = u_4 . W$, etc.

It remains now to throw these relations into a geometrical garb and incidentally to state the cross-connections of these forms in a variety of shapes.

If l is a λ line with constant coefficients a_1/a_2 l is a conic. This conic will contain three points independent of the choice of l. The proof of this lies in the identity, L, M, N denoting arbitrary λ points

$$\begin{vmatrix} a_1 L & a_2 L & l L \\ a_1 M & a_2 M & l M \\ a_1 N & a_2 N & l N \end{vmatrix} = a_1 a_2 l . L M N$$

when it is taken into consideration that the three conics

$$\begin{vmatrix} a_1 L & a_1 M & a_1 N \\ a_2 L & a_2 M & a_2 N \end{vmatrix} \text{ have three points in common.}$$

The three points thus defined will be written (a_1/a_2). They are situate on W, because $W \equiv a_1/a_2/a_3$.

The two triplets of points (a_1/a_2) and (a_1/a_3) are coresidual on W. More accurately stated, the two conics

$$a_1/a_2/l \text{ and } a_1/a_3/l$$

intersect W, besides in (a_1/a_2) and (a_1/a_3), in 3 identical points for brevity denoted by (l). This is evident from the identical relation

$$a_1/a_2/l \,.\, a_1/a_3/m - a_1/a_3/l \,.\, a_1/a_2/m = a_1/a_2/a_3 \,.\, a_1/l/m$$

where m is any λ line. For $a_1/a_2/a_3$ is $\equiv W$ and m may be so determined that $a_1/a_2/l$ and $a_1/a_2/m$ intersect in no more than three points on W.

To each λ point L correspond x lines $a_1 L$, $a_2 L$. To a line of λ points corresponds a pencil of x lines. Hence the ∞^2 pairs of lines $a_1 L$, $a_2 L$, where L is variable, represent a linear transformation of the x plane, that may be briefly denoted by (1, 2). If l is a λ line, a_1/l denotes a pencil of x lines, whose vertex may be (a_1/l). (a_1/l) and (a_2/l) are x points corresponding to each other by virtue of (1, 2). For if L and M are points on l, $a_1 L$ and $a_1 M$ intersect in (a_1/l), and $a_2 L$ and $a_2 M$ intersect in (a_2/l).

The triplet (a_1/a_2) is the self corresponding triangle of (1, 2). Indeed if P is such a point that it corresponds to itself by virtue of (1, 2), then a λ line l must exist so that

$$a_1/l \equiv P, \quad a_2/l \equiv P.$$

Hence if we compose in the x plane

$$a_1 P \equiv l \equiv a_2 P$$

and $a_1 P/a_2 P/m = 0$ however the λ line m may be chosen. Consequently P is one of the three points $a_1/(a_1/a_2)$.

The conic $a_1/a_2/l$ contains the two points (a_1/l) and (a_2/l). Denoting by L, M two points on l, we have $LM = l$, and

$$a_1 L \,.\, a_2 M - a_1 M \,.\, a_2 L = a_1/a_2/L M.$$

The conic contains therefore the point of intersection of $a_1 L$ and $a_1 M$. It contains, besides, the point of intersection of $a_1 L$ and $a_2 L$, i. e. of any two lines corresponding by virtue of (1, 2) whose corresponding λ point is situated on l.

Consider now the three triplets (a_1/a_2), (a_1/a_3), (a_2/a_3). Any three conics through them, $a_1/a_2/l$, $a_2/a_3/l$, $a_3/a_1/l$, that intersect W in the same 3 residual points, intersect besides in three points [$a_1/a_2/l$ and $a_1/a_3/l$ in (a_1/l), $a_1/a_2/l$ and $a_2/a_3/l$ in (a_2/l), $a_2/a_3/l$ and $a_1/a_3/l$ in (a_3/l)] which correspond to each other by virtue of the correspondences (1, 2), (1, 3) and (2, 3). This argument and result is in nowise restricted to any special cubics or special coresidual triplets on them. However we select three coresidual triplets on a cubic, the conics through them intersecting the cubic in an identical triplet of points intersect besides in three points which are correspondents in two linear transformations of the plane.

We shall now place ourselves in the viewpoint of neglecting all of the preceding equations except such as refer to the a_1, a_2, a_3, a_4 and W. And we shall give our attention to the study of those properties of these forms as apply to any cubic W. Let W be given and let a linear transformation, or collineation of its plane whose self-corresponding triangle (a_1/a_2) is on W, be arbitrarily selected and called (1, 2). Then any conic through (a_1/a_2) and a corresponding pointpair intersects W in a triplet (l) and conversely any conic through (a_1/a_2) contains just one corresponding pointpair, as easily is shown by elementary considerations. An auxiliary λ plane may then be constructed and forms a_1, a_2 calculated.

Let now a point P be arbitrarily selected on W. Through (a_1/a_2) we construct a conic $a_1/a_2/l$ that intersects W in a residual triplet (l). This conic also contains the point (a_1/l). Through (l), P and (a_1/l) a conic is determined, that we call $a_1/a_3/l$ and which intersects W, besides in (l) and P, in two points which with P form a triplet called (a_1/a_3). If similarly another point Q is arbitrarily chosen on W then a triplet (a_2/a_3) may be similarly determined so as to contain Q. And now the whole set a_1, a_2, a_3 and the transformations (1, 2), (1, 3), (2, 3) are fixed, because conics through (a_1/a_2), (a_1/a_3) and (a_2/a_3) intersecting W in an identical triplet (l) have their fourth point of intersection in corresponding points.

If the two points whose free choice led to the determination of a_3 are varied while everything else remains the same, then only such λ lines a_3' will be generated as are linearly dependent upon a_1, a_2, a_3. Indeed $a_1/a_2/a_3$ will be, according to the above construction, $\equiv W$, the cubic under discussion. Also $a_1/a_2/a_3' $ will be $\equiv W$. Constants b and b' must therefore exist so that $a_1/a_2/(b\,a_3 - b'\,a_3') = 0$. Let $b\,a_3 - b'\,a_3' = \gamma$. From $a_1/a_2/\gamma = 0$ it follows, if L, M, N are λ points

$$\begin{vmatrix} a_1 L & a_1 M & a_1 N \\ a_2 L & a_2 M & a_2 N \\ \gamma L & \gamma M & \gamma N \end{vmatrix} = 0.$$

But
$$\begin{vmatrix} a_1 L & a_1 M \\ a_2 L & a_2 M \end{vmatrix}, \quad \begin{vmatrix} a_1 M & a_1 N \\ a_2 M & a_2 N \end{vmatrix}, \quad \begin{vmatrix} a_1 L & a_1 N \\ a_2 L & a_2 N \end{vmatrix}$$

have three points in common. Hence γM must contain the 4th point common to the two first determinant conics and, γ being a straight line, it may easily be shown that γ must be a linear combination, with constant coefficients, of a_1 and a_2.

The relation between the 6 triplets on $W(a_1/a_2) \ldots (a_3/a_4)$ is then that they are coresidual and that the 6 conics through these 6 triplets and any residual triplet on W intersect in only 4 more points. We have shown above that this statement necessarily involves the linear dependence of a_1, a_2, a_3, a_4 expressed by (9). Equation (9) may find expression in a geometrical shape as follows: L being any λ point, $a_1 L$, $a_2 L$, $a_3 L$, $a_4 L$ therefore x lines, from

$$b_1 \,.\, a_1 L + b_2 \,.\, a_2 L + b_3 \,.\, a_3 L + b_4 \,.\, a_4 L = 0$$

it is evident that $b_1 . a_1 L + b_2 . a_2 L$ is the line passing through the intersections of $a_1 L / a_2 L$ and through that of $a_3 L / a_4 L$. Now to each L corresponds a definite $b_1 . a_1 L + b_2 . a_2 L$, and $b_1 . a_1 + b_2 . a_2$ is therefore in conjunction with a_1 a symbol for a definite linear transformation of the x plane, whose self-corresponding triangle must be on W. In fact this triangle is no other than the three points common to $a_1 / b_1 a_1 + b_2 a_2 / l$, i. e. no other than the triplet (a_1/a_2). Hence by some linear transformation T whose self-corresponding triangle is (a_1/a_2), each $a_1 L$ is converted into the line joining the intersections of $a_1 L / a_2 L$ and $a_3 L / a_4 L$. And a similar statement holds for the other diagonals of the quadrangle of corresponding lines $a_1 L$, $a_2 L$, $a_3 L$, $a_4 L$.

We shall now consider the forms u_i and v_i that were hitherto neglected. If L is any λ point, $v_1 v_2 L$, $v_2 v_3 L$, $v_3 v_1 L$ are three quartics that have 12 points in common; for it is clear that $v_1 v_2 L$ and $v_2 v_3 L$ contain the 4 points of the pencil $v_2 L$ and have their other common points on $v_1 v_3 L$. From the identity

$$v_1 v_2 v_3 \,.\, L = v_1 v_2 L \,.\, v_3 + v_2 v_3 L \,.\, v_1 + v_3 v_1 L \, v_2$$

it follows, since $v_1 v_2 v_3 = u_4 . W$, that these 12 points common to $v_1 v_2 L$, $v_2 v_3 L$, $v_3 v_1 L$ either lie on W or on u_4. Now $v_1 v_2 = a_4 u_3 - a_3 u_4$, hence $v_1 v_2 L$ intersects u_4 in the 9 points, where u_3 cuts that curve and besides in the 3 points of intersection of $a_4 L$ and u_4. Consequently $v_1 v_2 L$, $v_2 v_3 L$, $v_3 v_1 L$ have three of their common points on u_4, nine on W. $v_1 v_2 \equiv v_1 (b_2 v_1 - b_1 v_2) \equiv v_1 (a_3/a_4)$ intersects W always in the 3 points (a_3/a_4). Consequently the 9 variable points of the intersection of $v_1 v_2 L$ and W are those that are common to $v_1 v_3 L$ and $v_2 v_3 L$; and, by the same reasoning, also to $v_1 v_4 L$, $v_2 v_4 L$, $v_3 v_4 L$.

$v_1 v_2 L$ and $v_3 v_4 L$ have, beside these 9 points, still 7 points in common.

These complete with the 18 points (u_1, u_2), (u_3, u_4) the base of 25 points of a pencil of quintics. Indeed the two quintics

$$u_1 . v_1 L M + u_2 . v_2 L M$$

and

$$u_1 . v_1 L N + u_2 . v_2 L N$$

contain the 9 points (u_1, u_2), also those of (u_3, u_4) owing to identity (1), and besides 7 points obviously situate on the result of eliminating u_1, u_2 from above,

$$\begin{vmatrix} v_1 L M & v_2 L M \\ v_1 L N & v_2 L N \end{vmatrix} = v_1 v_2 L . L M N.$$

By the same reasoning and by means of the equation (1) these 7 points are also situate on $v_3 v_4 L$. Moreover, it is clear that none of these points will generally lie on say $v_1 v_3 L$. The group of 7 points is therefore characterized as that part of the intersection of $v_1 v_2 L$ and $v_3 v_4 L$ that is residual to the group of 9 points common to the $v_i v_j L$.

The 7 points $(v_1 v_2 L, v_3 v_4 L)$ form with the point of intersection of $a_1 L$, $a_2 L$ and with that of $a_3 L$, $a_4 L$ the base of a pencil of cubics. Since $u_1 . v_1 L M + u_2 . v_2 L M$ as well as $v_3 v_4 L = a_2 L . u_1 - a_1 L . u_2$ contain the 7 points, so will $a_1 L . v_1 L M + a_2 L . v_2 L M$. Hence this cubic and $a_1 L . v_1 L N + a_2 L . v_2 L N$ contain the 7 points. The two cubics contain, besides, point $(a_1 L, a_2 L)$, and, owing to relations (7) and (8), also $(a_3 L, a_4 L)$.

From all this it is then apparent: Let S be a linear system of cubics which has 4 linearly independent cubics (u_1, u_2, u_3, u_4) as base and which, besides, is not of that particular nature that the 18 points (u_1, u_2), (u_3, u_4) should admit more than 3 linearly independent quintics through them. Then two quintics through two groups of nine points common to two cubics of S, for inst. (u_1, u_2) and (u_3, u_4), will intersect in 7 residual points that with each one of the above group of nine points will lie on *one* quartic. The two quartics thus determined will have 9 residual points in common, and these 9 points will always lie on the same cubic W. The cubic thus determined belongs to the system S. Moreover, it is clear from what precedes, that however the selection of the original two pairs of cubics in S and of the two quintics through the pair of nine points may be varied the group of nine points determining W must always be one of a determinate system Γ of ∞^2 such nine-point groups, as any particular nine-point group of Γ does not depend at all on the choice of the two pairs of cubics (variety of choice obviously subjecting the v_1, v_2, v_3, v_4 only to a linear transformation of each other) but entirely on the choice of the two quintics through the 18 points, i. e. on the λ line l.

The cubic W of the system S has still other remarkable properties easily derived from the set of equations previously deduced. Thus (11) shows that each quintic

$$u_1 (b_4 . v_1 l — b_1 v_4 l) + W . v_4 l$$

also contains the 9 points (u_2, u_3). But $b_4 . v_1 l—b_1 . v_4 l$ has, wherever l may be situated, the 3 points (a_2/a_3) in common with W. Hence, a cubic (u_1) of S intersects W in 9 points, that with any nine points common to two other cubics of S (u_2, u_3) determine a triplet of points (a_2/a_3) such that any quintic through above 18 points also contains the triplet. And this triplet is situate on W, and all triplets thus determined, by varied choice of the cubics (u_1, u_2, u_3) in S, are coresidual. Moreover, each such triplet is residual to any one of the nine-point groups of Γ, and a quartic $(v_1 v_4 l)$ containing any one such triplet and an individual of Γ contains also nine points of intersection of two cubics of S (u_2, u_3).

A group of nine points common to two cubics of S is determined when two of its points are arbitrarily given, for a cubic of S may be made to pass through any three points, and two points determine therefore a pencil in S. The ∞^4 system Δ of nine points forming the base of a pencil in S, and the system Γ of nine point groups has this relationship that any two individuals of Δ and Γ lie on a quartic. Let for inst. a nine point group of Δ be the intersection of u'_1, u'_2. Choosing as base of S u'_1, u'_2, u'_3, u'_4 (any 4 linearly independent cubics two of which are u'_1, u'_2) it is clear from the preceding, that the corresponding v'_i are such as to yield quartics $v'_3 v'_4 L$ containing, according to the selection of L, any given individual of Γ and (u'_1, u'_2).

Not every quartic through an individual of Γ contains an individual of Δ. Let a given individual Γ_L of Γ be common to $v_1 v_2 L$, $v_1 v_3 L \dots . v_3 v_4 L$. Then any quartic through Γ_L will be

$$f = \xi_{1,2}\, v_1 v_2 L + \dots . + \xi_{3,4}\, v_3 v_4 L$$

where $\xi_{1,2} \dots . \xi_{3,4}$ are 6 indeterminates. Should f contain an individual of Δ it must be representable in the shape $v'_i\, v'_j\, L$, where v'_i and v'_j are linearly dependent upon the v_i. This, as in the case of straight lines in space, requires an equation to be satisfied, namely $\xi_{1,2} . \xi_{3,4} + \xi_{2,3} . \xi_{1,4} + \xi_{3,1} . \xi_{2,4} = 0$.

The same method leads to interesting results when 4 curves of order n are under discussion. And it is a suggestive fact that the connections between 4 forms of order $(n-1)$, $(n-2)$, etc., recur again at the higher orders. In the case $n=3$ that has just been studied, for instance, the equation

$$a_1 M . v_1 l + a_2 M . v_2 l + a_3 M . v_3 l + a_4 M . v_4 l = 0$$

where M is a variable point on l, shows that the a_1/l, a_2/l, a_3/l, a_4/l are pencils of lines that stand to the conics $v_1 l$, $v_2 l$, $v_3 l$, $v_4 l$ in the same report as the pencils previously named g_1, g_2, g_3, g_4 to the 4 conics of the case $n=2$. And the analogy goes very much further. So is the triangle EFG in the case $n=3$ represented by the triangle (l) on W, and all the other points and lines that were mentioned in the configuration belonging to $n=2$ find ready interpretation for the set of conics $v_2 l$.

If the condition referring to the 18 points (u_1, u_2), (u_3, u_4) is not satisfied, more than three linearly independent quintics will exist containing the 18 points and the v_1, v_2, v_3, v_4 will then be points in a λ space of manifoldness 4. It follows then

$$v_1 v_2 v_3 = u_4 . W, \text{ etc.,}$$

where W is a λ plane containing v_1, v_2, v_3, v_4 and is of order 3 in the x coordinates or else vanishes. Again we have the equations

$$v_1 v_2 = a_4 u_3 - a_3 u_4, \text{ etc.,}$$

where the a_i are of 1st order in the x_i, and of the dimension of a line in the λ space. Obviously

$$(a_4 . u_3 - a_3 . u_4)(a_4 . u_3 - a_3 . u_4) = 0$$

i. e. $a_4 . u_3 - a_3 . u_4$ composed with itself is 0. Hence

$$a_4 a_4 . u_3^2 - 2 a_4 a_3 . u_4 u_3 + a_3 a_3 . u_4^2 = 0.$$

Consequently $a_4 a_4$ is divisibly by u_4. But $a_4 a_4$ is only of order 2 in the x_i. Therefore

$$a_4 a_4 = 0,\ a_3 a_3 = 0,\ a_4 a_3 = 0.$$

Considered as λ forms, $v_1 v_2$, $a_4 a_3$ are three lines having a common point of intersection. The same argument holds in respect to v_1, v_3 and a_2, a_4, etc. The v_1, v_2, v_3, v_4 must therefore be λ points in a λ plane W in which also a_1, a_2, a_3, a_4 are situated.

From $v_1 v_2 = a_4 u_3 - a_3 u_4$ it follows that $v_1 a_4$ is divisible by u_4. Let be

$$v_1 a_4 = b_1 . u_4$$

b_1 will then be (unless it vanishes) a λ space independent of the x plane. We shall also have $v_1 a_3 = b_1 . u_3$ and generally $v_i a_j = b_i . u_j$ if $i \ddagger j$.

To find $v_i a_i$, we compose

$$v_1 v_2 = a_4 u_3 - a_3 u_4$$

with v_3.

$$u_4 . W = b_3 u_4 . u_3 - v_3 a_3 u_4$$

or

$$v_3 a_3 = b_3 . u_3 - W.$$

Composing $u_1 v_1 . + \dots . + u_4 . v_4$ with a_1 we obtain

$$b_1 u_1 + b_2 u_2 + b_3 u_3 + b_4 u_4 = W.$$

$b_1 v_1$ is $= 0$. Consequently

$$b_2 v_1 . u_2 + b_3 v_1 . u_3 + b_4 v_1 . u_4 = 0.$$

This shows $b_2 v_1 = 0$, $b_3 v_1 = 0$, $b_4 v_1 = 0$. The b_i *must* vanish, and therefore also W. The v_i will all be on the same λ line a and the a_i will be, as λ lines, congruent with a. From this $a_i = c_i . a$, where c_i is a numeric. Therefore

$$v_1 v_2 = (c_4 u_3 - c_3 u_4) . a, \text{ etc.},$$
$$v_1 (c_2 v_2 + c_3 v_3 + c_4 v_4) = 0$$

and

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0.$$

Let $a = x_1 \alpha_1 + x_2 \alpha_2 + x_3 \alpha_3$, where x_1, x_2, x_3 are the x_i variables, $\alpha_1, \alpha_2, \alpha_3$ lines in the λ space independent of x. Now $(v_1 v_2)(v_1 v_2) = 0$. Hence $aa = 0$, and it follows $\alpha_i \alpha_j = 0$. The α_i are intersecting lines. They cannot lie in the same λ plane P, or $P v_i = 0$, and the v_1, v_2, v_3, v_4 would be forms of a plane, not of a space of 4 manifoldness, contrary to hypothesis. Consequently the α_i must have a λ point G in common.

Let then be $a = \alpha G$, where α is a λ point. $v_i G$ is $\equiv a$, therefore let be

$$v_i G = p_i . a$$

where p_i is a linear x form. Then

$$v_i G = p_i . \alpha G$$

and

$$v_i = p_i . \alpha + q_i . G.$$

The original identity, after the insertion of these values, gives then the two equations

$$p_1 u_1 + p_2 u_2 + p_3 u_3 + p_4 u_4 = 0$$
$$q_1 u_1 + q_2 u_2 + q_3 u_3 + q_4 u_4 = 0$$

where the p_i are of the 1st order, the q_i of the 2nd order. From this, as before,

$$p_1 q_2 - p_2 q_1 = c_4 u_3 - c_3 u_4, \text{ etc.},$$
$$c_1 p_1 + c_2 p_2 + c_3 p_3 + c_4 p_4 = 0$$
$$c_1 q_1 + c_2 q_2 + c_3 q_3 + c_4 q_4 = 0$$

Also
$$(c_1 u_2 - c_2 u_1) p_2 + (c_1 u_3 - c_3 u_1) p_3 + (c_1 u_4 - c_4 u_1) p_4 = 0$$
$$(c_1 u_2 - c_2 u_1) q_2 + \dots\dots\dots\dots = 0$$

This can only be, if the three cubics $c_1 u_2 - c_2 u_1$, $c_1 u_3 - c_3 u_1$, $c_1 u_4 - c_4 u_1$ have 7 points in common.

The salient fact then is, that whenever the 18 points (u_1, u_2), (u_3, u_4) are such as to permit more than three linearly independent quintics through them, some identity $p_1 u_1 + p_2 u_2 = -p_3 u_3 - p_4 u_4$ will exist and therefore a quartic will contain all the points. When the above condition is satisfied, a quintic form f will exist apolar to u_1, u_2, u_3, u_4. And conversely *any* quintic f being given, 4 cubics u_1, u_2, u_3, u_4 may be found that are apolar to f. The 7 points common to $c_1 u_2 - c_2 u_1$, etc., may be denoted by $A \dots G$. These 7 points will be those by means of which f may be represented as sum of seven fifth powers of linear forms $A^5 + \dots + G^5$. We need not dwell on this point that is easily made plain but by considerations foreign to the subject of this paper.

Now much of what has been said and done here may be generalized. A form of order $2n-1$ being arbitrarily given, the set of forms of order n apolar to it may be studied in an analogous fashion. Nor is this calculation restricted in any way in the number of variables x_i.

The calculation may be generalized in a somewhat different direction and yields, without much labor, an interesting and very applicable result. As a starting point we use the theory of my paper "Zur Theorie der Moduln und Ideale." Accordingly the reader will be supposed to understand the meaning of the form $\Omega\,(u_1 \dots u_m)$ and to be acquainted with propositions I, II, III of that paper.

Let $u_1 \dots u_m$ be m forms of m homogenous variables, whose resultant does not vanish. Their orders may be designated by $n_1 \dots n_m$. A form of order

$n_1 + \dots + n_m - m$ will then exist apolar to all of them. This form may be written Ω. There will be no form of order $n_1 + \dots + n_m - m$ other than Ω apolar to $u_1 \dots u_m$, and no form of higher order will be apolar to the set (Proposition III). Besides Ω will have this remarkable property, not hitherto laid down, that any form of lower order than Ω apolar to $u_1 \dots u_m$ must necessarily be a polar form of Ω. With other words, if ψ is a form apolar to $u_1 \dots u_m$, of order $n_1 + \dots + n_m - m - N$, where N is a positive integer, then a form g of order N will exist such that identically

$$\psi = g \times \Omega$$

i. e. that ψ is the polar of g in respect to Ω.

To show this, let F be a form of order $n_1 + \dots + n_m - m - N$, f one of order N, and let all the coefficients of F and f be indeterminate quantities. The magnitude $F.f \times \Omega$ is then a bilinear form F of the indeterminate coefficients of F and f. If it is required that $F.f \times \Omega = 0$ while the set of coefficients of F remains arbitrary, then the set of coefficients of f will be subject to a certain number α of conditions which remains the same, if F and f change their roles in this. This is merely an expression of one of the elementary properties of bilinear forms.

It follows then that the number of conditions imposed upon a form of order $n_1 + \dots + n_m - m - N$, to be apolar to Ω is the same as the corresponding number for forms of order N. And the number of contragredient forms of order $n_1 + \dots + n_m - m - N$ apolar to $u_1 \dots u_m$ has the same value. Let this number be denoted by α.

Exactly α forms $g_1 \dots g_\alpha$ of order N will then exist, which are linearly independent and no linear combination of which belongs to the module $(u_1 \dots u_m)$, or, what is the same, is apolar to Ω. Let g represent an indeterminate form of this linear system. Then $g \times \Omega$ will never vanish, and will contain α indeterminatae. But the linear system of forms of order $n_1 + \dots + n_m - m - N$ apolar to $u_1 \dots u_m$ has the manifoldness α, and there is only one such system if two forms congruent modulo $(u_1 \dots u_m)$ are for this purpose considered equivalent or identical. Consequently the set $g \times \Omega$ represents that system; and if ψ is a form of order $n_1 + \dots + n_m - m - N$ apolar to $u_1 \dots u_m$ then a form g will indeed exist so that $\psi = g \times \Omega$.

This theorem immediately leads to any number of geometrical propositions. Let us apply it, for inst., to the proposition concerning the 18 points (u_1, u_2),

(u_3, u_4) through which more than 3 linearly independent quintics pass. A quintic ψ will then exist apolar to u_1, u_2, u_3, u_4. The Ω of u_1, u_2, u_3 is of order $3+3+3-3=6$. ψ being a quintic apolar to u_1, u_2, u_3 a linear form g exists such that $\psi = g \times \Omega$. But $u_4 \times \psi = 0$, consequently $u_4 \cdot g \times \Omega = 0$, and $u_4 \cdot g$ belongs to the modulus u_1, u_2, u_3. This shows at once that the 18 points (u_1, u_2), (u_3, u_4) are on a quartic.

If as a particular case 16 of the 18 points are the base of a pencil of quartics — when indeed the supposition will be satisfied — then the present proposition shows itself to be identical with the one announced previously.

It would naturally not at all be difficult to draw similar conclusions for curves of higher orders or for forms of higher manifoldness.

In this point the proposed method of calculating geometric configurations and the general theory of moduli meet. In all probability the connection between the two disciplines will grow much more intimate as the method will further develop. It is clear that equations of the type $\Sigma u_i \cdot v_i = 0$ with their λ space adjoined are very apt to express the properties of systems of forms of certain orders which belong to two different modules simultaneously, which, for inst., contain two distinct irreducible or reducible geometric configurations. And from the preceding examples it is fairly evident that the proposed treatment of such equations yields results.

It is true that only a few and perhaps rather simple examples have been discussed in what precedes. But the author may be excused if he points out that it would have been easy enough to extend the method much further and that the difficulty for him consisted rather in limiting the examples to such as would bring out, in a lucid and easy manner, some of the characteristic properties of the proposed method.

NEW YORK, *May* 3, 1906.